THE JOINING PROBLEM OF NONSTEADY HEAT TRANSFER IN A THICK-WALLED CHANNEL

K. M. Shlyakhtina

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We present an analytical solution for the problem of nonsteady heat transfer in the motion of a hot gas in a flat channel with walls of semiinfinite thickness.

In studying the flow of a hot gas in a channel we are confronted with the problem of determining the temperature field within the channel wall and the related heat losses from the gas moving through the channel. This problem reduces to the simultaneous solution of the two-dimensional equation of nonsteady heat conduction and the equation which gives the change in the enthalpy of the gas. Problems of this type – referred to as joining problems – have been studied in detail in [1, 2], in their application to the steady-state case.

In a number of practical cases we should take into consideration the thermal nonsteadiness, since consideration of the study-state problem does not yield satisfactory results.

Solution of such a problem in general formulation is possible with numerical methods and the use of a digital computer. The numerical solution calls for considerable expenditure of time and provides no opportunity of an operative analysis of the effect of changes in the original parameters on the final result. A number of approximation methods have therefore been developed for the solution of this kind of problem, and these make it possible – given certain assumptions – to achieve an analytical solution. The simplifications usually reduce either to the assumption [3] that the channel wall is heated uniformly through the thickness (whether for a flat channel or one of circular cross section) or to the assumption [4] that the δ/R radius is small (for the circular channel), thus making it possible to limit ourselves to the second term in the series for the Laplace transforms. Such simplifications do not always correspond to the physical conditions of the problem. When the channel wall is very thick, we must take into account the temperature profile within the wall, and this leads to the need for a numerical solution.

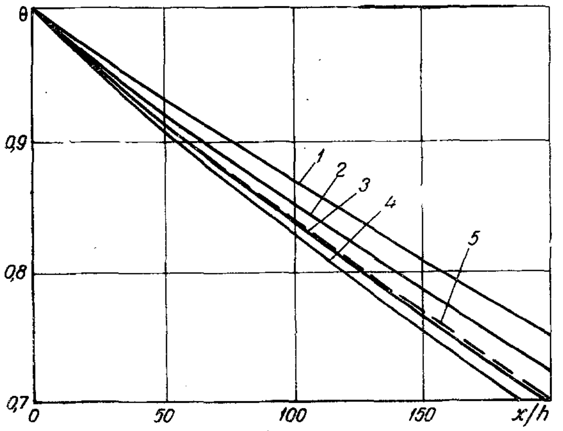

Fig. 1. Temperature distribution for the gas along the channel at a fixed instant of time (t = 30 sec): 1) δ = 3 mm; 2) 5; 3) 10; 4) 20; 5) semiinfinite wall.

Below we propose the treatment of a thick wall as a semiinfinite wall. The problem can then be formulated in the following manner. A gas with a constant temperature at the channel inlet is flowing at a constant velocity through a flat channel whose width is 2h and whose walls are semiinfinite in thickness. We can neglect the longitudinal conduction of heat because it is small in comparison with the transverse. The coefficient of convection heat transfer along the channel is assumed to be constant.

The system of equations in this case has the following form:

$$\frac{\partial\theta(x,\ t)}{\partial t}+w\,\frac{\partial\theta(x,\ t)}{\partial x}=-\frac{\alpha}{c_p\rho gh}\left[\theta(x,\ t)-T(x,\ 0,\ t)\right],\quad (1)$$

$$\frac{\partial T(x,\ y,\ t)}{\partial t}=a\,\frac{\partial^2 T(x,\ y,\ t)}{\partial y^2}$$

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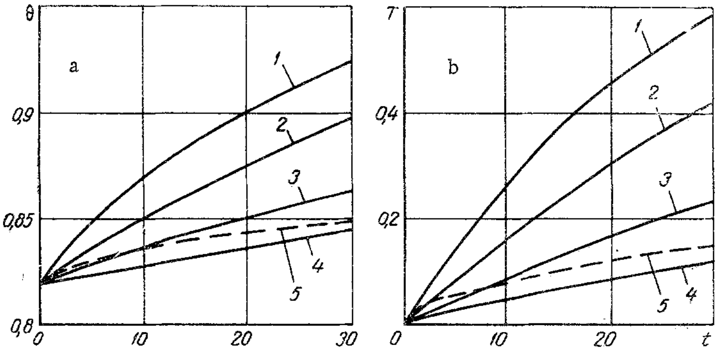

Fig. 2. Time variation in gas temperature (a) and in wall-surface temperature (b) at a fixed section of the tube (x/h = 100): 1-5) see Fig. 1.

with the following initial and boundary conditions:

$$\begin{gathered} t=0 \quad \theta=T=T_0, \\ t\geqslant 0 \quad x=0,\ \theta=\theta_0, \\ \lambda\left.\frac{\partial T}{\partial y}\right|_{y=0}=-\alpha\left[\theta(x,\ t)-T(x,\ 0,\ t)\right], \\ \left.\frac{\partial T}{\partial y}\right|_{y=\infty}=0. \end{gathered} \tag{2}$$

We introduce the dimensionless variables

$$\bar{\theta}(x,\ t)=\frac{\theta(x,\ t)-T_0}{\theta_0-T_0};\quad \bar{T}(x,\ y,\ t)=\frac{T(x,\ y,\ t)-T_0}{\theta_0-T_0}. \tag{3}$$

The form of Eqs. (1) thus remains as before (the bar over the dimensionless variables is dropped), and the boundary conditions are written as follows:

$$\begin{gathered} t=0 \quad \theta=T=0, \\ t>0 \quad x=0,\ \theta=1, \\ \lambda\left.\frac{\partial T}{\partial y}\right|_{y=0}=-\alpha\left[\theta-T(x,\ 0,\ t)\right], \\ \left.\frac{\partial T}{\partial y}\right|_{y=\infty}=0. \end{gathered} \tag{4}$$

Using the Laplace transforms

$$\theta_1=s\int_0^\infty\theta\exp(-st)\,dt,$$

$$T_1=s\int_0^\infty T\exp(-st)\,dt,$$

we derive the system of equations for the images

$$\begin{gathered} a\frac{\partial^2 T_1}{\partial y^2}=sT_1, \\ s\theta_1+w\frac{d\theta_1}{dx}=-k\left[\theta_1-T_1(x,\ 0)\right], \\ \left.\frac{dT_1}{dy}\right|_{y=0}=-H\left[\theta_1-T_1(x,\ 0)\right], \\ \left.\frac{dT_1}{dy}\right|_{y=\infty}=0,\quad \theta_1|_{x=0}=1, \end{gathered} \tag{5}$$

where $k=\alpha/c_p\rho g$; $H=\alpha/\lambda$, and whose solution has the form

TABLE 1. Values of the First Terms in Series (9)

t, sec	1st term	2nd term	3rd term	4th term
1	1	0,014610	0,000085	0,000000
3	1	0,031536	0,000398	0,000003
10	1	0,043457	0,000761	0,000008
3	1	0,070662	0,002043	0,000033

$$\theta_1 = \exp\left[\frac{kx}{w}\left(\frac{H}{\sqrt{\frac{s}{a}}+H}-1\right)-\frac{sx}{w}\right], \quad T_1 = \theta_1 \frac{H}{\sqrt{\frac{s}{a}}+H}\exp\left(-\sqrt{\frac{s}{a}}\,y\right). \tag{6}$$

To find the Laplace reconversions, let us examine the expansion of the function F(s) in series:

$$F(s) = \exp\left[\frac{kx}{w}\frac{1}{\sqrt{\frac{s}{aH^2}}+1}\right] = 1+\frac{kx}{w}\frac{1}{\sqrt{\frac{s}{aH^2}}+1} + \frac{1}{2}\left(\frac{kx}{w}\right)^2\left(\frac{1}{\sqrt{\frac{s}{aH^2}}+1}\right)^2 + \frac{1}{6}\left(\frac{kx}{w}\right)^3\left(\frac{1}{\sqrt{\frac{s}{aH^2}}+1}\right)^3 + \ldots \tag{7}$$

With a small value for time ($s \to \infty$) this series converges rather rapidly as a consequence of the multiplier $1/(\sqrt{s/aH^2}+1)$ independently of the value of kw/w (kw/w is finite). In the most interesting case in which time is not small, this multiplier is close to 1. Consequently, series (7) will converge rapidly only when $kx/w < 1$. Converting the multiplier kx/w to clearer form, we obtain

$$\frac{kx}{w} = \mathrm{St}\,\frac{x}{h} = \mathrm{St}\,\frac{l}{h}\,\frac{x}{l}. \tag{8}$$

When $\mathrm{St}(l/h) < 1$ series (7) converges rapidly even in the worst limit case ($s \to 0$). We can the limit ourselves to several of the first terms in series (7) and we can find the Laplace reconversions for the function F(s).

It should be noted that the above-cited condition is capable of satisfaction over a rather broad range of variation in the physical parameters.

We will use the theorems of operational calculus so that the formulas for the gas temperature in the channel and the temperature of the wall can finally be written in the form

$$\begin{aligned}\theta(x,t) = \exp\left(-\mathrm{St}\frac{x}{h}\right)&\left\{\eta\left(t-\frac{x}{w}\right)+\mathrm{St}\frac{x}{h}\left[1-\exp\left[aH^2\left(t-\frac{x}{w}\right)\right]\right.\right.\\ &\times\left.\mathrm{erfc}\left[H\sqrt{a\left(t-\frac{x}{w}\right)}\right]\right]+\frac{1}{2}\left(\mathrm{St}\frac{x}{h}\right)^2\left[\eta\left(t-\frac{x}{w}\right)\right.\\ &+\left(2H^2a\left(t-\frac{x}{w}\right)-1\right)\mathrm{erfc}\left[H\sqrt{a\left(t-\frac{x}{w}\right)}\right]\exp\left[H^2a\left(t-\frac{x}{w}\right)\right]\\ &-\left.\frac{2}{\sqrt{\pi}}H\sqrt{a\left(t-\frac{x}{w}\right)}\right]+\frac{1}{6}\left(\mathrm{St}\frac{x}{h}\right)^3\left\{\eta\left(t-\frac{x}{w}\right)\right.\\ &-\left[2H^4a^2\left(t-\frac{x}{w}\right)^2-H^2a\left(t-\frac{x}{w}\right)+1\right]\exp\left[H^2a\left(t-\frac{x}{w}\right)\right]\\ &\times\mathrm{erfc}\left(H\sqrt{a\left(t-\frac{x}{w}\right)}\right)+\left.\left.\frac{2H\sqrt{a\left(t-\frac{x}{w}\right)}}{\sqrt{\pi}}\left[H^2a\left(t-\frac{x}{w}\right)-1\right]\right\}+\ldots\right\},\end{aligned} \tag{9}$$

$$\begin{aligned}T(x,y,t) = \int_0^t \theta(x,\tau)&\left\{H\sqrt{\frac{a}{\pi(t-\tau)}}\exp\left(-\frac{y^2}{4a(t-\tau)}\right)\right.\\ &-\left.aH^2\exp\left[yH+aH^2(t-\tau)\right]\mathrm{erfc}\left[\frac{y}{2\sqrt{a(t-\tau)}}+H\sqrt{a(t-\tau)}\right]\right\}d\tau,\end{aligned} \tag{10}$$

where

$$\eta\left(t-\frac{x}{w}\right)=\begin{cases}0, & t<\dfrac{x}{w},\\ 1, & t\geqslant\dfrac{x}{w}.\end{cases}$$

The numerical calculations show that for an approximate evaluation of the heat losses on the part of the gas we can use (9) in the form

$$\theta(x,\ t)=\exp\left(-\mathrm{St}\frac{x}{h}\right)\left\{\eta\left(t-\frac{x}{w}\right)+\mathrm{St}\frac{x}{h}\left[1-\exp\left[aH^2\left(t-\frac{x}{w}\right)\right]\mathrm{erfc}\left[H\sqrt{a\left(t-\frac{x}{w}\right)}\right]\right]\right\}, \tag{11}$$

and here the error does not exceed 5-7%, which is fully acceptable for engineering calculations.

It is easy to demonstrate that at a constant gas temperature solution (10) reduces to the familiar solution of the problem of nonsteady heat conduction for a semiinfinite wall with boundary conditions of the third kind. As an example, we calculated the temperature distribution for the gas along a flat channel, as well as the temperature profile in a wall, with the following initial data:

$$\mathrm{St}=0.002;\ H=10\ \mathrm{m}^{-1};\ l/h=200;\ a=0.04\ \mathrm{m}^2/\mathrm{h}. \tag{12}$$

For purposes of comparison we present the calculation results for these same parameters under identical heating conditions in the assumption of a uniform temperature profile in the wall. As is well known, in this case these quantities are determined by the following formulas:

$$\theta(x,\ t)=\eta\left(t-\frac{x}{w}\right)\mathrm{St}\frac{x}{h}\frac{aH}{\delta}\int_0^{t-\frac{x}{w}}\exp\left(-\frac{Ha\tau}{\delta}\right)\frac{I_1\left(2\sqrt{H\,\mathrm{St}\frac{x}{h}\frac{a\tau}{\delta}}\right)}{\sqrt{H\frac{a\tau}{\delta}\mathrm{St}\frac{x}{h}}}d\tau\exp\left(-\mathrm{St}\frac{x}{h}\right),$$
$$T(x,\ y,\ t)=\frac{Ha}{\delta}\exp\left(-\frac{Hat}{\delta}\right)\int_0^t\theta(x,\ \tau)\exp\left(\frac{Ha\tau}{\delta}\right)d\tau. \tag{13}$$

The calculation was performed for walls with thicknesses of δ = 3, 5, 10, and 20 mm. The results of the gas-temperature calculation and those of the wall-surface temperature are shown in Figs. 1 and 2.

As was to be expected, with a sufficiently thin wall (δ = 3 and 5mm) the heat losses from the gas in the channel are lower than in the case of a semiinfinite wall. However, in this example, even with δ = 20 mm, the assumption of uniform heating yields an understated value for the gas temperature. It is clear that for any real wall (of finite thickness) this temperature must be higher than in a channel with semiinfinite walls, so that consequently the assumption of uniform heating for a rather thick wall will yield clearly understated results.

In this example of calculation the parameter $\mathrm{St}(l/h)$ is equal to 0.4. Here, limiting ourselves to the first terms of the series in expansion (7) leads to a very slight error. For clarity, Table 1 gives the values of the first four terms of the series at various instants of time for x/h = 200.

NOTATION

$\theta(x, t)$	is the gas temperature at the point with the coordinate x at the instant of time t;
$T(x, y, t)$	is the wall temperature at the point with the coordinates x and y at the instant of time t;
λ, a	are the coefficients of thermal conductivity and thermal diffusivity for the wall material;
w, ρ, c_p	are, respectively, the velocity, the density, and the heat capacity of the gas;
α	is the coefficient of heat transfer between the gas and the wall;
h	is the half-width of the channel;
l	is the channel length;
St	is the Stanton number;
x	is the longitudinal coordinate;
y	is the coordinate normal to the surface of the wall;
δ	is the wall thickness.

LITERATURE CITED

1. T. L. Perel'man, Inzh.-Fiz. Zh., 4, No. 4, 5, 8 (1961).
2. A. V. Luikov and T. L. Perel'man, in: Heat and Mass Transfer [in Russian], Vol. 6, Nauka i Tekhnika, Minsk (1966).
3. I. A. Kuznetsov, Inzh.-Fiz. Zh., No. 11 (1964).
4. Inuie Haiyasi, Journal of Heat Transfer, Transactions of the American Society of Engineers [Russian translation], No. 4 (1965).
5. G. P. Ivantsov and B. L. Lyubov, Dokl. Akad. Nauk SSSR, 35, No. 2 (1952).
6. A. V. Luikov, The Theory of Heat Conduction [in Russian], GITTL, Moscow (1952).